Instructions. (30 points) Solve each of the following problems.

(1^{pts}) **1.** $\lim_{x\to -1} \dfrac{x^4-1}{x-1} =$

(a) -4 (b) Does Not Exist

(c) 4 (✔) 0

Solution:

$$\begin{aligned}\lim_{x\to -1} \frac{x^4-1}{x-1} &= \frac{(-1)^4-1}{-1-1}\\ &= \frac{1-1}{-2}\\ &= \frac{0}{-2} = 0\end{aligned}$$

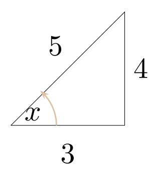
(1^{pts}) **2.** If $\cos x = \dfrac{3}{5}$, $\dfrac{3\pi}{2} < x < 2\pi$ then $\sin x =$

(a) $\dfrac{-3}{4}$ (b) $\dfrac{-4}{3}$

(c) $\dfrac{-5}{4}$ (✔) $\dfrac{-4}{5}$

Solution:

Since $\cos x = \dfrac{3}{5} = \dfrac{\text{adjacent}}{\text{hypotenuse}}$ we draw a triangle is shown below: Now, $H^2 = A^2 + O^2 \Rightarrow 25 = 9 + O^2 \Rightarrow O = 4$. Since $\dfrac{3\pi}{2} < x < 2\pi$, then x lies in the forth quadrant. Hence since the $y-$ axis is negative, then $\sin x < 0$ and since the $x-$axis is positive then $\cos x > 0$. Hence $\sin x = \dfrac{-4}{5}$.



(1^{pts}) **3.** $\lim_{x\to\infty} \dfrac{x-4x^3}{2x^3-1} =$

(a) $\dfrac{-1}{2}$ (b) ∞

(c) 0 (✔) -2

Solution:

Since $\lim_{x\to\infty}(x-4x^3) = -\infty$, $\lim_{x\to\infty}(2x^3-1) = \infty$ we have I.F. type $-\infty/\infty$. Divide each term in the numerator and each term in the denominator by the highest power in the denominator.

$$\lim_{x\to\infty}\frac{x-4x^3}{2x^3-1}=\lim_{x\to\infty}\frac{\frac{x-4x^3}{x^3}}{\frac{2x^3+1}{x^3}}=\lim_{x\to\infty}\frac{\frac{x}{x^3}-\frac{4x^3}{x^3}}{\frac{2x^3}{x^3}-\frac{1}{x^3}}=\lim_{x\to\infty}\frac{\frac{1}{x^2}-4}{2-\frac{1}{x^3}}=\frac{0-4}{2-0+0}=-2$$

(1pts) **4.** $\sin\left(\frac{7\pi}{18}\right)\cos\left(\frac{\pi}{18}\right)-\sin\left(\frac{\pi}{18}\right)\cos\left(\frac{7\pi}{18}\right)=$

(a) $\frac{1}{2}$ (b) 2

(✔) $\frac{\sqrt{3}}{2}$ (d) 0

Solution:

$$\sin\left(\frac{7\pi}{18}\right)\cos\left(\frac{\pi}{18}\right)-\sin\left(\frac{\pi}{18}\right)\cos\left(\frac{7\pi}{18}\right)=\sin\left(\frac{7\pi}{18}-\frac{\pi}{18}\right)==\sin\left(\frac{\pi}{3}\right)=\frac{\sqrt{3}}{2}.$$

(1pts) **5.** The curve $f(x)=\frac{x^2-2x-3}{x^3-9x}$ has a vertical asymptote at

(a) $x=0,\quad x=\pm3$ (b) $y=0,\quad y=-3$

(c) $x=0,\quad x=3$ (✔) $x=0,\quad x=-3$

Solution:

Write $f(x)=\frac{(x-3)(x+1)}{x(x-3)(x+3)}$. The zeroes of the denominator are -3, 0, and 3. To check that $x=3$ is a vertical asymptote or not we take the limit at 3 from both sides. $\lim_{x\to3}f(x)=\lim_{x\to3}\frac{\cancel{(x-3)}(x+1)}{x\cancel{(x-3)}(x+3)}=\lim_{x\to3}\frac{x+1}{x(x+3)}=\frac{4}{18}=\frac{2}{9}$. Hence $x=3$ is not a vertical asymptote.

To check that $x=-3$ we take the limit $\lim_{x\to-3^+}f(x)=\lim_{x\to-3^+}\frac{\overset{+}{(x-3)(x+1)}}{\underset{+}{x(x-3)(x+3)}}=\infty$. Hence $x=-3$ is a vertical asymptote. To check that $x=0$ we take the limit $\lim_{x\to0^+}f(x)=$

$\lim\limits_{x\to 0^+} \dfrac{(x-3)^{-}(x+1)}{x(x-3)^{-}(x+3)} = \infty$. Hence $x = 0$ is a vertical asymptote. Thus the function has vertical asymptote at $x = 0$, and $x = -3$.

(1^{pts}) **6.** If

$$f(x) = \begin{cases} x+2, & \text{if } x < -3; \\ 2, & \text{if } x = -3; \\ \dfrac{x^3+27}{x^2-9}, & \text{if } x > -3. \end{cases} \quad \text{Then } \lim_{x\to -3^+} f(x) =$$

(a) $\frac{9}{2}$ (b) $-\frac{3}{2}$

(c) 0 (✔) $-\frac{9}{2}$

Solution:

Note that when computing limit as $x \to a^+$ means you approaches a from the right side that is $x > a$.

$$\begin{aligned}
\lim_{x\to -3^+} f(x) &= \lim_{x\to -3^+} \frac{x^3+27}{x^2-9} \\
&= \lim_{x\to -3^+} \frac{x^3+27}{x^2-9} && \text{A direct substation will give us I.F. } 0/0 \\
&= \lim_{x\to -3^+} \frac{(x+3)(x^2-3x+9)}{(x+3)(x-3)} && \text{Factoring } x+3 \text{ from denominator and numerator} \\
&= \lim_{x\to -3^+} \frac{\cancel{(x+3)}(x^2-3x+9)}{\cancel{(x+3)}(x-3)} \\
&= \lim_{x\to -3^+} \frac{x^2-3x+9}{x-3} \\
&= \frac{(-3)^2-3(-3)+9}{-3-3} \\
&= \frac{27}{-6} = -\frac{9}{2}.
\end{aligned}$$

(1^{pts}) **7.** $\dfrac{d^{42}}{dx^{42}}(\cos x) =$

(a) $-\sin x$ (b) $\sin x$

(c) $\cos x$ (✔) $-\cos x$

Solution:

Since $42 = 4(10) + 2$ then $\dfrac{d^{42}}{dx^{42}}(\cos x) = \dfrac{d^2}{dx^2}(\cos x) = \dfrac{d}{dx}(-\sin x) = -\cos x$.

(1^{pts}) **8.** If $\dfrac{x^2+2x}{x} \le f(x) \le 3x+2, \quad x \in [-1,1], x \ne 0,$ then $\lim\limits_{x\to 0} f(x) =$

(a) 0 (✔) 2

(c) -2 (d) 1

Solution:

Since $\lim\limits_{x\to 0} \dfrac{x^2+2x}{x} = \lim\limits_{x\to 0} \dfrac{x(x+2)}{2x} = \lim\limits_{x\to 0}(x+2) = 2$, and $\lim\limits_{x\to 0}(3x+2) = 2$, then by The Sandwich Theorem we have $\lim\limits_{x\to 0} f(x) = 2$.

(1^{pts}) **9.** $\lim_{x\to 0} \frac{\tan^3(2x)}{x^3} =$

(a) $\frac{1}{8}$

(b) 2

(✔) 8

(d) $\frac{1}{2}$

Solution:

$$\begin{aligned} \lim_{x\to 0} \frac{\tan^3(2x)}{x^3} &= \lim_{x\to 0} \left(\frac{\tan(2x)}{x}\right)^3 && \tfrac{a^n}{b^n} = \left(\tfrac{a}{b}\right)^n. \\ &= \lim_{x\to 0} \left(\frac{2\tan(2x)}{2x}\right)^3 && \text{make the top similar to the angle} \\ &= \left(2\lim_{x\to 0} \frac{2x}{\tan(2x)}\right)^3 && \text{Use that } \lim_{x\to 0}\frac{\tan x}{x} = 1 = \lim_{x\to 0}\frac{x}{\tan x} \\ &= (2.1)^3 = 8. \end{aligned}$$

(1^{pts}) **10.** $\lim_{t\to 0} \frac{\tan(2t)\sin t}{t^2} =$

(a) $\frac{1}{2}$

(✔) 2

(c) 1

(d) -2

Solution:

Direct substation will give us I.F. type 0/0.

$$\begin{aligned} \lim_{t\to 0} \frac{\tan(2t)\sin t}{t^2} &= \lim_{t\to 0} \frac{\tan(2t)}{t}\frac{\sin t}{t} \\ &= 2\lim_{t\to 0} \frac{\tan(2t)}{2t}\frac{\sin t}{t} && \text{use } \lim_{\theta\to 0}\frac{\tan\theta}{\theta} = 1 = \lim_{\theta\to 0}\frac{\sin\theta}{\theta} \\ &= 2(1)(1) = 2 \end{aligned}$$

(1^{pts}) **11.** If $y = \frac{\cos x}{1-\sin x}$, then $y' =$

(a) $\frac{\sin x}{1-\sin x}$

(✔) $\frac{1}{1-\sin x}$

(c) $\frac{-\cos x}{(1-\cos x)^2}$

(d) $\frac{\cos x}{1-\sin x}$

Solution:

$$
\begin{aligned}
y' &= \frac{(1-\sin x)(-\sin x)-(\cos x)(-\cos x)}{(1-\sin x)^2} \\
&= \frac{-\sin x+\sin x\sin x+\cos x\cos x}{(1-\sin x)^2} \\
&= \frac{-\sin x+\sin^2 x+\cos^2 x}{(1-\sin x)^2} && \text{Use the fact } \cos^2 x+\sin^2 x=1 \\
&= \frac{1-\sin x}{(1-\sin x)^2} && \text{simplify} \\
&= \frac{1}{1-\sin x}.
\end{aligned}
$$

(1^{pts}) **12.** The graph of the function $F(x) = x^3 - 27x$, has a horizontal tangent line at

(✔) $x = \pm 3$ (b) $x = 3$

(c) $x = -3$ (d) $x = 0$

Solution:

$$
\begin{aligned}
F(x) &= x^3 - 27x \\
F'(x) &= 3x^2 - 27 = 3(x^2 - 9) \\
F'(x) &= 0 \\
3(x^2-9) &= 0 \\
x &= \pm 3.
\end{aligned}
$$

(1^{pts}) **13.** An equation for the tangent line to the curve $f(x) = x^3 + x$ at $x = 1$ is

(a) $y = -4x - 2$ (✔) $y = 4x - 2$

(c) $y = 4x + 2$ (d) $y = -4x + 6$

Solution:

The slope of the tangent line to $f(x) = x^3 + x$ at $x = 1$ is $f'(1)$.

$$f(x) = x^3 + x \Rightarrow f'(x) = 3x^2 + 1.$$

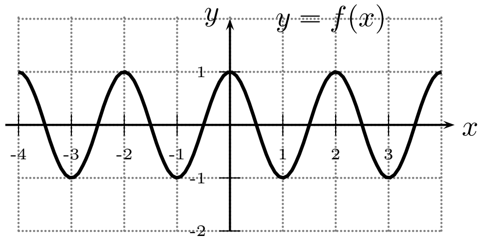
Hence

$$\text{the slope of the tangent } = f'(1) = 4.$$

Also $f(1) = (1)^3 + (1) = 2$. Now, we have $m = 4$ and $(1, 2)$, hence

$$y - y_1 = m(x - x_1) \Rightarrow y - 2 = 4(x - 1) \Rightarrow y - 2 = 4x - 4 \Rightarrow y = 4x - 2.$$

(1^{pts}) **14.** The accompanying figure shows the graph of $y = f(x)$. Then the period of $y = f(x)$ is



(a) 1

(b) 4

(c) 2π

(✔d) 2

Solution:

From the graph we can see that the function repeat itself every 2 units. Hence the period is 2

(1^{pts}) **15.** If $f(1) = 3$, then $\lim_{x\to 1} f(x)$ must exist.

(a) True

(✔b) False

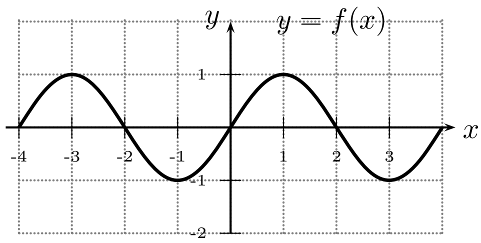
Solution:

False. Let $f(x) = \begin{cases} 1, & \text{if} x > 1; \\ 3, & \text{if} x = 1; \\ -1, & \text{if} x < 1. \end{cases}$ Then $f(1) = 3$, but $\lim_{x\to 1^+} f(x) = 1 \neq -1 = \lim_{x\to 1^-} f(x)$, Hence $\lim_{x\to 1} f(x)$ does not exist.

(1^{pts}) **16.** The accompanying figure shows the graph of $y = f(x)$. Then $f'(-2) < f'(0)$.

(✔a) True

(b) False



Solution:

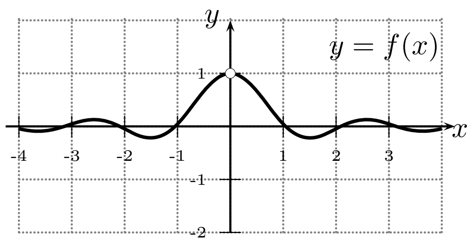
From the graph we can see that the tangent line to the graph of $y = f(x)$ at -2 is falling down (left to right) and hence its slope is negative. Thus, since the first derivative is the slope of the tangent line to the graph at -2 then $f'(-2) < 0$.

From the graph we can see that the tangent line to the graph of $y = f(x)$ at 0 is rising up (left to right) and hence its slope is positive. Thus, since the first derivative is the slope of the tangent line to the graph at 0 then $f'(0) > 0$. Now, $f'(-2) < 0 < f'(0)$.

(1^{pts}) **17.** The accompanying figure shows the graph of $y = f(x)$. Then $\lim_{x\to 0} f(x) =$

(a) 0 (✔b) 1

(c) Does Not Exist (d) -1



Solution:

since $\lim_{x\to 0^-} f(x) = 1 = \lim_{x\to 0^+} f(x)$, then $\lim_{x\to 0} f(x) = 1$

(1pts) **18.** $\lim_{x\to 3} \dfrac{x-3}{2-\sqrt{x+1}} =$

(a) 4

(b) $\dfrac{-1}{4}$

(c) 0

(d)✔ -4

Solution:

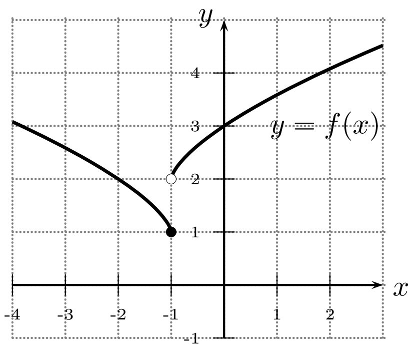
A direct substation will give us I.F. 0/0. Now,

$$\begin{aligned}
\lim_{x\to 3} \frac{x-3}{2-\sqrt{x+1}} &= \lim_{x\to 3} \frac{x-3}{2-\sqrt{x+1}} \cdot \frac{2+\sqrt{x+1}}{2+\sqrt{x+1}} && \text{Multiply by } 1 = \frac{2+\sqrt{x+1}}{2+\sqrt{x+1}} \\
&= \lim_{x\to 3} \frac{(x-3)(2+\sqrt{x+1})}{2^2-(\sqrt{x+1})^2} \\
&= \lim_{x\to 3} \frac{(x-3)(2+\sqrt{x+1})}{4-(x+1)} && \text{Simplify} \\
&= \lim_{x\to 3} \frac{\cancel{(x-3)}(\sqrt{x+6}+3)}{-\cancel{(x-3)}} \\
&= \lim_{x\to 3} [-(2+\sqrt{x+1})] \\
&= -[2+\sqrt{3+1}] \\
&= -4.
\end{aligned}$$

(1pts) **19.** The accompanying figure shows the graph of $y = f(x)$. Then f is differentiable at $x = -1$.

(a) True

(b)✔ False



Solution:

f is not differentiable at $x = -1$ since the the function is discontinuous at $x = -1$.

(1pts) **20.** If $y = x \sin x \csc x$, then $y' =$

(a)✔ 1

(b) -1

(c) 0

(d) $-\cos x \csc x \cot x$

Solution:

$$\begin{aligned}
y &= x \sin x \csc x \\
y &= x \sin x \frac{1}{\sin x} \\
y &= x \\
y' &= 1.
\end{aligned}$$

(1pts) **21.** The function $f(x) = \sqrt{2 - |x|}$ is continuous on

(a) $(-\infty, -2] \cup [2, \infty)$ (✔b) $[-2, 2]$

(c) $(-\infty, 2) \cup (2, \infty)$ (d) $[2, \infty)$

Solution: Notice that $f(x) = \sqrt{2 - |x|}$ is an even root function, then it is continuous on its domain which is the set of real number such that $2 - |x| \geq 0$. Now

$$2 - |x| \geq 0 \Leftrightarrow -|x| \geq -2$$
$$\Leftrightarrow |x| \leq 2 \quad \Leftrightarrow -2 \leq x \leq 2.$$

Hence f is continuous on $[-2, 2]$.

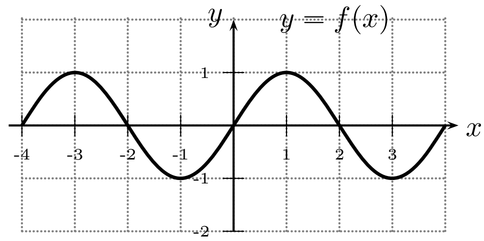
(1pts) **22.** The accompanying figure shows the graph of $y = f(x)$. Then $f'(1) =$

(✔a) 0

(b) -1

(c) Undefined

(d) 1



Solution:

From the graph we can see that the tangent line to the graph of $y = f(x)$ at 1 is horizontal ($y = 1$) and hence its slope is zero. Now, since the first derivative is the slope of the tangent line to the graph at 2 then $f'(1) = 0$.

(1pts) **23.** If $y = x^2 + \dfrac{1}{x^3} - \sqrt{5}$, then $y''' =$

(a) $\dfrac{60}{x^6} + \dfrac{1}{2\sqrt{5}}$ (✔b) $\dfrac{-60}{x^6}$

(c) $\dfrac{60}{x^6}$ (d) $\dfrac{-60}{x^6} - \dfrac{1}{2\sqrt{5}}$

Solution:

$$y = x^2 + \frac{1}{x^3} - \sqrt{5}$$
$$y = x^2 + x^{-3} - \sqrt{5}$$
$$y' = 2x - 3x^{-4}$$
$$y'' = 2 + 12x^{-5}$$
$$y''' = -60x^{-6} = \frac{-60}{x^6}.$$

(1^{pts}) **24.** If $y = \dfrac{3x+1}{x-1}$, then $y' =$

(a) $\dfrac{2}{(x-1)^2}$

(b) $\dfrac{-2}{(x-1)^2}$

(c) $\dfrac{6x-4}{(x-1)^2}$

(✔) $\dfrac{-4}{(x-1)^2}$

Solution:

$$y' = \frac{\overbrace{(x-1)}^{\text{Bottom}}\ \overbrace{(3x+1)'}^{\text{Derivative of Top}} - \overbrace{(3x+1)}^{\text{Top}}\ \overbrace{(x-1)'}^{\text{Derivative of Bottom}}}{\overbrace{(x-1)^2}^{\text{Bottom}^2}}$$

$$= \frac{(x-1)(3) - (3x+1)(1)}{(x-1)^2}$$

$$= \frac{3x-3-(3x+1)}{(x-1)^2}$$

$$= \frac{3x-3-3x-1)}{(x-1)^2}$$

$$= \frac{-4}{(x-1)^2}.$$

(1^{pts}) **25.** $\lim\limits_{\theta \to 0} \cos\left(\dfrac{\pi\theta}{\sin(\theta)}\right) =$

(✔) -1

(b) 0

(c) 1

(d) ∞

Solution:

$$\begin{aligned}\lim_{\theta\to 0} \cos\left(\frac{\pi\theta}{\sin(\theta)}\right) &= \cos\left(\lim_{\theta\to 0}\frac{\pi\theta}{\sin(\theta)}\right)\\ &= \cos\left(\frac{\pi\theta}{\sin(\theta)}\right)\\ &= \cos(\pi)\\ &= -1.\end{aligned}$$

(1^{pts}) **26.** If $y = x\sin x$, then $y' =$

(a) $\cos x$

(b) $\sin x + 1$

(c) $\sin x - x\cos x$

(✔) $\sin x + x\cos x$

Solution:

$$\begin{aligned}y' &= (x)'\sin x + x(\sin x)'\\ &= \sin x + x(\cos x)\\ &= \sin x + x\cos x.\end{aligned}$$

(1 pts) **27.** $\lim\limits_{x\to 3} \dfrac{x^2-9}{x^2-x-6} =$

(a) $\dfrac{5}{6}$

(b) ✔ $\dfrac{6}{5}$

(c) $\dfrac{-6}{5}$

(d) 0

Solution:

$$\begin{aligned}
\lim_{x\to 3} \frac{x^2-9}{x^2-x-6} &= \lim_{x\to 3} \frac{x^2-9}{x^2-x-6} && \text{A direct substation will give us I.F. } 0/0 \\
&= \lim_{x\to 3} \frac{(x+3)(x-3)}{(x-3)(x+2)} && \text{Factoring } x-3 \text{ from denominator and numerator} \\
&= \lim_{x\to 3} \frac{(x+3)\cancel{(x-3)}}{\cancel{(x-3)}(x+2)} \\
&= \lim_{x\to 3} \frac{x+3}{x+2} \\
&= \frac{3+3}{3+2} = \frac{6}{5}.
\end{aligned}$$

(1 pts) **28.** $\lim\limits_{x\to -2^-} \dfrac{x^2-1}{4x+8} =$

(a) ∞

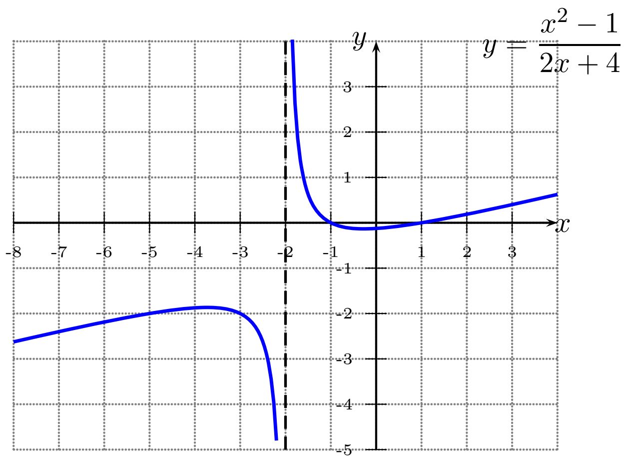
(b) 0

(c) Does Not Exist

(d) ✔ $-\infty$

Solution:

$$\begin{aligned}
\lim_{x\to -2^-} \frac{x^2-1}{4x+8} &= \lim_{x\to -2^-} \frac{x^2-1}{2x+4} && \text{A direct substation gives I.F. } 3/0. \\
&= \lim_{x\to -2^-} \frac{-x-1}{x+2} && \text{If } x<-2 \text{ and near } -2\text{,then } x^2-1>0,\ 4x+8<0. \\
&= \lim_{x\to -2^-} \frac{\overset{+}{x^2-1}}{\underset{-}{4x+8}} = -\infty
\end{aligned}$$



(1^{pts}) **29.** The curve $f(x) = \dfrac{x - 2x^3 + 1}{x^3 - x + 5}$ has a horizontal asymptote at

(✔) $y = -2$ (b) $y = 0$

(c) $y = 5$ (d) $x = -2$

Solution:

To find the horizontal asymptote we take the limit as $x \to \pm\infty$ Note that both the numerator and the denominator $\to \pm\infty$, as $x \to \pm\infty$. To find this limit divided both the numerator and the denominator by the highest power of x in the denominator which is x^3. So,

$$\begin{aligned}\lim_{x\to\pm\infty} \frac{x - 2x^3 + 1}{x^3 - x + 5} &= \lim_{x\to\pm\infty} \frac{\dfrac{x - 2x^3 + 1}{x^3}}{\dfrac{x^3 - x + 5}{x^3}} \\ &= \lim_{x\to\pm\infty} \frac{\dfrac{1}{x^2} - 2 + \dfrac{1}{x^3}}{1 - \dfrac{1}{x^2} + \dfrac{5}{x^3}} \\ &= \frac{0 - 2 + 0}{1 - 0 + 0} = -2.\end{aligned}$$

Therefore $y = -2$ is a horizontal asymptote.

(1^{pts}) **30.** $\displaystyle\lim_{x\to 1} \frac{|x-3| - 2}{x - 1} =$

(a) 1 (b) Does Not Exist

(c) 0 (✔) -1

Solution:

A direct substation gives I.F. 0/0. Note that if $x > 1$ or $x < 1$ near(close) to 1 then $x - 3 < 0 \Rightarrow |x - 3| = -(x - 3)$.

$$\begin{aligned}\lim_{x\to 1} \frac{|x-3| - 2}{x - 1} &= \lim_{x\to 0} \frac{-(x-3) - 2}{x - 1} \\ &= \lim_{x\to 1} \frac{-x + 3 - 2}{x - 1} \\ &= \lim_{x\to 1} \frac{-x + 1}{x - 1} \\ &= \lim_{x\to 1} \frac{-(x-1)}{x - 1} \\ &= \lim_{x\to 1} (-1) \\ &= -1\end{aligned}$$